ATTENUATION OF FORCED WEAK PLANE PRESSURE WAVES IN A GAS WITH RADIATIVE HEAT TRANSFER

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This paper considers the effect of radiative heat transfer on the propagation of forced plane harmonic pressure waves of small amplitude in an infinite emitting-absorbing inviscid nonconducting gas. The radiative pressure and radiative energy are neglected. The purpose of this paper is: a) to construct a theory based on the exact directional distribution of the total (frequency-integrated) specific intensity and to use this theory to calculate the parameters of the wave motion, b) to compare the exact theory with results obtained on the basis of the direction-averaged equation of radiative transfer [1] so as to estimate the errors introduced by various directional approximations and to demonstrate the importance of the anisotropy of radiation in radiation gasdynamics.

In the linear theories of Stokes, Rayleigh, Kirchhoff, and Langevin the problem of wave attenuation is separated into special cases, in each of which only one single process is considered. This separation is admissible when to the first approximation the effects of the different dissipation mechanisms (viscosity, thermal conductivity, radiation, etc.) are additive. When only one factor is considered the problem becomes much simpler and the results are more amenable to physical interpretation, and these results can then be used in the solution of the complete problem.

§1. The characteristic equation. The one-dimensional plane motion (in the x-direction) of a compressible inviscid fluid with heat transfer by emission and absorption of radiant energy is described by the system of equations of radiation gasdynamics, which consists of the equations of continuity, momentum, energy, and radiative transfer together with the equation of state (for a two-parameter gas) and the radiative equation of state—Kirchhoff's law (if local thermodynamic equilibrium is assumed),

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{d\rho}{dt} = -\rho\frac{\partial u}{\partial x},$$

$$\frac{dU}{dt} = -\frac{1}{\rho}\left(p\frac{\partial u}{\partial x} + \frac{\partial H}{\partial x}\right),$$

$$\cos\vartheta\,\frac{\partial J}{\partial x} = \omega(B - J), \quad U = \varphi(\rho, T),$$

$$p = f(\rho, T), \quad B = \frac{\eta}{\alpha} = \frac{\sigma'}{\pi}T^4,$$

$$H(x, t) = 2\pi\int_0^\pi J(x, t, \vartheta)\cos\vartheta\sin\vartheta\, d\vartheta, \qquad \omega \equiv \rho\alpha. \quad (1.1)$$

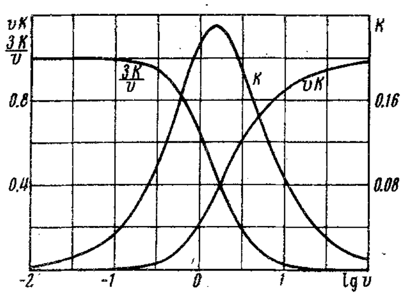

Fig. 1. Reduced damping coefficients of nearly adiabatic pressure waves.

Here p, ρ, T, u are the pressure, density, temperature, and velocity of the fluid, x is the coordinate, t is time, U is the internal energy density, J is the total specific intensity of radiation, ϑ is the angle between the ray of radiation and the x axis, H is the radiative flux, α is the mass absorption coefficient, η is the integrated emission coefficient, and σ' is the Stefan-Boltzmann constant. All variables in these equations are considered to be continuous.

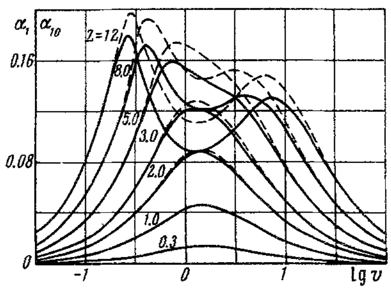

Fig. 2. Damping coefficients per wavelength α_1 (solid lines) and per wavelength of the adiabatic sound wave α_{10} (dashed lines) for $\gamma = 5/3$. The value of Z is shown on each curve.

Consider an infinite homogeneous gas at rest (whose parameters we shall denote by subscript 0) and let small plane harmonic perturbations be excited in the plane x = 0. As a result, all the parameters of the gas will assume the perturbed values

$$R(x, t) = R_0[1 + R'(x, t)],$$

$$R'(x, t) = R'(0, 0)\exp(ax + i\sigma t),$$

$$u = c_0 u', \quad H = 2\pi B_0 H',$$

$$u_0 = 0, \quad H_0 = 0, \quad J_0 = B_0, \qquad (1.2)$$

where the primes denote perturbations. Here R denotes any of the variables p, ρ, T, ω, J, B; c_0 is the adiabatic speed of sound, a is a complex constant, to be determined from the solution, and σ is the circular frequency of the forced oscillations. Assuming that all perturbations and their derivatives are small and substituting (1.2) into the linearized form of (1.1), one obtains a system of linear homogeneous equations for the perturbations. The condition for the existence of nontrivial solutions of this system yields the characteristic equation

$$\frac{1}{2q}\ln\frac{1+q}{1-q} = 1 + \gamma i\zeta_1\frac{1+m^2}{\gamma+m^2},$$

$$q = \frac{a}{\omega_0} = mv \equiv q_r + iq_i, \quad m \equiv m_r + im_i = \frac{c_0 a}{\sigma},$$

$$v = \frac{1}{w} = \frac{\sigma}{c_0\omega_0}, \quad \zeta_1 = \zeta^{-1} = \frac{v}{Z}, \quad Z = \frac{16\pi B_0}{(\rho c_v T)_0 c_0},$$

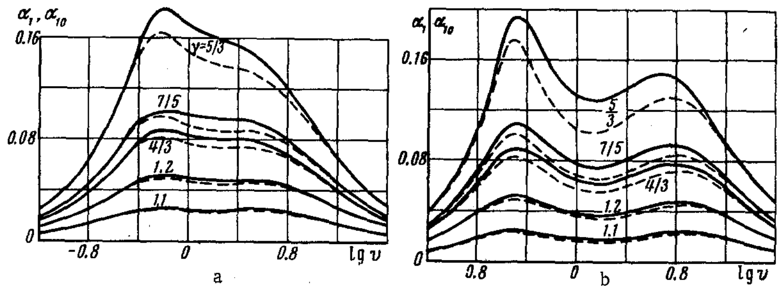

Fig. 3. Damping coefficients α_1 (solid lines) and α_{10} (dashed lines) for different values of the ratio of specific heats (indicated on the curves): a) Z = 6, b) Z = 10.

$$c_0 = \left(\gamma h_1 \frac{p_0}{\rho_0}\right)^{1/2}, \quad \gamma = 1 + \frac{h_2 h_3}{h_1 h_4} + \frac{h_2}{h_1 h_4}\frac{p_0}{\rho_0}, \quad h_1 = \left(\frac{\partial \ln p}{\partial \ln \rho}\right)_0,$$

$$h_2 = \left(\frac{\partial \ln p}{\partial \ln T}\right)_0, \quad h_3 = -\left(\frac{\partial U}{\partial \rho}\rho\right)_0, \quad h_4 = (c_V T)_0. \quad (1.3)$$

Here γ is the ratio of specific heats and c_V is the specific heat at constant volume.

The logarithmic function in (1.3) replaces the integral

$$\ln\frac{1+q}{1-q} = \int\limits_{1-q}^{1+q}\frac{dz}{z} \quad (1.4)$$

where the path of integration in the complex plane is along the straight line 1 + q, 1 − q. Consequently, we take the branch of the logarithm with the argument in (0, π).

The analysis of the effect of radiative transfer on weak waves is reduced to the determination of q (m or a) from Eq. (1.3) as a function of the frequency and the properties of the gas. Both sides of the equation are even functions of m; the forced oscillations propagate in either direction according to the same rule. There exist no purely imaginary or real roots of the characteristic equation except the trivial solution corresponding to the gas at rest. Thus the solution represents damped traveling waves.

All unknown variables can be expressed in terms of one of them, e.g., the temperature perturbation,

$$p' = \frac{\gamma h_2}{\gamma + m^2}T', \quad \rho' = -\frac{h_2}{h_1}\frac{m^2}{\gamma + m^2}T',$$

$$u' = i\frac{h_2}{h_1}\frac{m}{\gamma + m^2}T', \quad B' = 4T',$$

$$U' = \left(h_4 + \frac{h_2 h_3}{h_1}\frac{m^2}{\gamma + m^2}\right)T',$$

$$J' = \frac{4T'}{1 + q\cos\vartheta}, \quad H' = \frac{2}{q}\left(1 - \frac{1}{2q}\ln\frac{1+q}{1-q}\right)T'. \quad (1.5)$$

The physical meaning of v, w, Z, ζ, m_r, m_i was explained in another paper [1]; the correspondence between the present notation and that of [1] is

$$w = g w^\circ, \quad Z = 2\frac{g'}{g}Z^\circ, \quad \zeta = 2g'\zeta^\circ, \quad Zv = 2\frac{g'}{g^2}Z^\circ v^\circ, \quad (1.6)$$

where the superscript ° denotes the variables of [1]. The symbols g and g' denote constant coefficients associated with the directional averaging of the equation of radiative transfer. The meaning of q is clear from the identities

$$|q_r| \equiv \xi = \alpha_\tau = |m_r| v = \lambda_r |a_r|, \quad \lambda_r = \frac{1}{\omega_0},$$

$$v = \frac{2\pi\lambda_r}{l_0} = \frac{2\pi}{l_{\tau 0}} = 2\pi n_{\tau 0}, \quad |q_i| \equiv \eta = |m_i| v =$$

$$= \frac{\sigma}{\omega_0 c_a} = v_a = \frac{1}{w_a} = \frac{2\pi\lambda_r}{l} = \frac{2\pi}{l_\tau} = 2\pi n_\tau, \quad (1.7)$$

Here α_τ is the wave damping coefficient per photon mean free path λ_r; $l_{\tau 0}$, l_τ are the optical thicknesses of an adiabatic sound wave and the pressure wave; $n_{\tau 0}$ and n_τ are the corresponding optical wave numbers.

In the following we determine the damping coefficients of the pressure waves: 1) α_{a0} per wavelength of a sound wave $l_0 = 2\pi c_0/\sigma$; 2) α_a per wavelength of the pressure wave; 3) α_{a1} per unit length; 4) α_2, proportional to the ratio of α_{a1} to the square of the frequency, which is of some interest in acoustics:

$$\alpha_{a1} = |a_r|, \quad \alpha_{a0} = 2\pi\alpha_{10} = 2\pi|m_r|,$$

$$\alpha_a = 2\pi\alpha_1 = 2\pi\frac{m_r}{m_i}, \quad \alpha_2 = \frac{|m_r|}{v}. \quad (1.8)$$

The characteristic equation contains the two governing dimensionless parameters v and Z, or combinations of these. These parameters can be expressed in terms of the characteristic time for wave oscillations —the period ϑ, the characteristic time for absorption t_{ra}—the time during which the wave travels over a distance equal to one radiation mean free path, and the characteristic time for emission t_{re}—the time required for the emission of the variable part of the internal energy,

$$\vartheta = \frac{2\pi}{\sigma}, \quad t_{ra} = \frac{\lambda_r}{c_0} = \frac{1}{c_0\omega_0}, \quad t_{re} = \frac{\varepsilon_0}{E} = \left(\frac{\rho c_v T}{4\pi B\omega}\right)_0,$$

$$E = 4\pi\eta_0, \quad \varepsilon_0 = h_4 = (c_v T)_0, \quad (1.9)$$

$$v = \frac{2\pi t_{ra}}{\vartheta}, \quad Z = \frac{4t_{ra}}{t_{re}}, \quad \zeta_1 = \frac{\pi t_{re}}{2\vartheta}, \quad Zv = \frac{8\pi t_{ra}}{\vartheta t_{re}}.$$

These parameters also admit interpretation in terms of energy,

$$Z = 4\frac{4\pi\eta_0 t_{ra}}{c_{v0}T_0}, \quad \zeta_1 = \frac{\pi}{2}\frac{c_{v0}T_0}{4\pi\eta_0\vartheta},$$

$$Zv = \frac{6\pi k_r T_0}{l_0(\rho c_v T c_0)_0}, \quad k_r = \frac{16\pi\lambda_r B_0}{3T_0}. \quad (1.10)$$

Here E is the thermal energy emitted by unit mass per unit time, ε_0 is the thermal internal energy of the gas, and k_r is the radiative conductivity. The number Z characterizes the ratio of the energy emitted by unit mass during the time t_{ra} to its thermal internal energy, ζ_1 characterizes the ratio of the thermal internal energy to the energy emitted by unit mass during one period, and the product Zv characterizes the ratio of the radiative-conductive heat flux to the convective heat flux. In all these parameters the speed of sound c_0 is used as the characteristic speed. It can be seen that Zv is the reciprocal (to within a constant multiplier) of the Peclet number based on k_r, c_0.

The physical interpretation of the results of the study of Eq. (1.3) can be given within the framework of the terminology of relaxation acoustics also from an energetic viewpoint.

§2. **Nearly adiabatic waves.** It follows from the characteristic equation that the necessary and sufficient condition for the existence of nearly adiabatic waves (weakly damped and propagating at almost the speed of sound) is that the parameter β be small, where

$$\beta = KZ, \qquad K = \frac{1}{v}\left(1 - \frac{1}{v}\operatorname{arc\,tg} v\right),$$

$$K_2 = \frac{\alpha_3}{2}\frac{v}{K}, \qquad \alpha_3 = 2\frac{\gamma - 1}{1 + v^2}. \qquad (2.1)$$

For small β and γ not too close to 1 we obtain

$$\pm m_r = \frac{\gamma-1}{2\gamma}\beta - \frac{\gamma-1}{2\gamma}\beta^3\left\{1 - \frac{K_2}{8\gamma^2}(3 + 9\gamma - \alpha_3) + \frac{K_2^2}{4\gamma^2}\right\} +$$

$$+ \frac{\gamma-1}{256\gamma^5}\beta^5\left\{128(3\gamma - 2)\gamma^3 + K_2\left[5(1 + 5\gamma + 19\gamma^2 - 153\gamma^3) -\right.\right.$$

$$\left. - (1 - 2\gamma - 159\gamma^2)\alpha_3 - 2\alpha_3^2\left(\frac{1}{3} + 13\gamma - \alpha_3\right)\right] +$$

$$+ 4K_2^2\left[5(1 + 6\gamma + 25\gamma^2) - 2(3 + 17\gamma)\alpha_3 + \frac{11}{3}\alpha_3^2\right] -$$

$$\left. - 8K_2^3[5(1 + 3\gamma) - 3\alpha_3] + 8K_2^4\right\} + O(\beta^7),$$

$$\pm m_i = 1 + \frac{\gamma-1}{8\gamma^2}\beta^2(1 + 3\gamma - 2K_2) + \frac{\gamma-1}{128\gamma^4}\beta^4\left\{1 + 5\gamma + 35\gamma^2 -\right.$$

$$- 105\gamma^3 + 4K_2\left[1 + 6\gamma + 41\gamma^2 - (1 + 7\gamma)\alpha_3 + \frac{2}{3}\alpha_3^2\right] -$$

$$\left. - 24K_2^2\left(1 + 3\gamma - \frac{1}{2}\alpha_3\right) + 8K_2^3\right\} + O(\beta^6),$$

$$\alpha_1 = \frac{\gamma-1}{2\gamma}\beta + \frac{\gamma-1}{16\gamma^3}\beta^3\{1 + 2\gamma - 11\gamma^2 + K_2(1 + 11\gamma - \alpha_3) - 2K_2^2\} +$$

$$+ \frac{\gamma-1}{256\gamma^5}\beta^5\left\{3 + 12\gamma + 20\gamma^2 - 462\gamma^3 + 555\gamma^4 + K_2[7 + 67\gamma +\right.$$

$$+ 293\gamma^2 - 1007\gamma^3 - (7 + 26\gamma - 193\gamma^2)\alpha_3 - 26\gamma\alpha_3^2 + 2\alpha_3^3] +$$

$$+ 4K_2^2\left[8\gamma + 154\gamma^2 - 2(1 + 19\gamma)\alpha_3 + \frac{11}{3}\alpha_3^2\right] -$$

$$\left. - 8K_2^3(3 + 17\gamma - 3\alpha_3) + 8K_2^4\right\} + O(\beta^7). \qquad (2.2)$$

The positive function K(v) has a single extremum— a maximum (Fig. 1) at the point $v = v_m$, where $v_m = 1.514994$ ($w_m = 0.660068$), $K(v_m) = 0.229878$. The parameter β satisfies $\beta \le 0.2299Z$, i.e., for any given frequency it is small if Z is small, which is the case in all gases and liquids in all states which are not extremely far removed from the normal state. The inequality $\beta \ll 1$ is satisfied for any given Z if v is either sufficiently small or sufficiently large, and in these two cases the inequality $\beta \ll 1$ becomes

$$1)\ v \ll 1, \quad Zv \ll 1; \qquad 2)\ v \gg 1, \quad \zeta \ll 1. \qquad (2.3)$$

To a first approximation (2.2) yields

$$\alpha_a \approx \alpha_{a0} = \pi\frac{\gamma-1}{\gamma}\beta, \qquad \alpha_\tau = \frac{\gamma-1}{\gamma}\beta v,$$

$$\alpha_2 = \frac{\gamma-1}{\gamma}\frac{\beta}{v}. \qquad (2.4)$$

The true damping coefficient reaches a maximum $\alpha_{amax} = 0.229878\,\pi(\gamma - 1)Z\gamma^{-1}$ at $v = v_m$, and tends to zero when v is very large or very small. The coefficient is a monotone function of v and increases with increasing v from 0 to $\frac{1}{2}(\gamma - 1)Z\gamma^{-1}$. The coefficient $\alpha_{a1} \sim \sigma^2$ only for $v \le {\sim}0.1$. The general behavior of the curves is the same as in [1], but there is considerable quantitative difference. All damping coefficients are ~Z and increase with increasing γ.

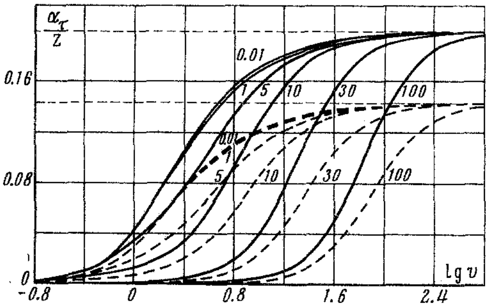

Fig. 4. Damping coefficient per radiation mean free path α_τ for $\gamma = 5/3$ (solid lines) and $\gamma = 7/5$ (dashed lines). The value of Z is indicated on each curve.

For small v the right sides of (2.2) take simpler forms,

$$\pm m_r = \frac{\gamma-1}{2\gamma}z_1\left(1 - \frac{3}{5}v^2 + \frac{3}{7}v^4\right) - \frac{\gamma-1}{16\gamma^3}z_1^3\left[5\gamma^2 - 30\gamma +\right.$$

$$\left. + 33 + \frac{3}{5}(-35\gamma^2 + 154\gamma - 143)v^2\right] + \frac{\gamma-1}{256\gamma^5}z_1^5(543\gamma^4 - 3012\gamma^3 +$$

$$+ 7794\gamma^2 - 9876\gamma + 4679) + O(k_7),$$

$$\pm m_i = 1 + \frac{\gamma-1}{8\gamma^2}z_1^2\left[7 - 3\gamma + \frac{6}{5}(5\gamma - 9)v^2\right] +$$

$$+ \frac{\gamma-1}{128\gamma^4}z_1^4(35\gamma^3 - 385\gamma^2 - 1001\gamma - 715) + O(k_6),$$

$$\alpha_1 = \frac{\gamma-1}{2\gamma}z_1\left(1 - \frac{3}{5}v^2 + \frac{3}{7}v^4\right) - \frac{\gamma-1}{8\gamma^3}z_1^3\left[\gamma^2 - 10\gamma + 13 -\right.$$

$$\left. - \frac{3}{10}(42\gamma^2 - 163\gamma + 145)v^2\right] + \frac{\gamma-1}{256\gamma^5}z_1^5(496\gamma^4 - 2502\gamma^3 +$$

$$+ 6314\gamma^2 - 8242\gamma + 4062) + O(k_7),$$

$$z_1 = \tfrac{1}{2}Zv, \qquad k_6 = z_1^2(v^4 + z_1^2v^2 + z_1^4),$$

$$k_7 = z_1(v^6 + z_1^2v^4 + z_1^4v^2 + z_1^6). \qquad (2.5)$$

The leading terms are in qualitative agreement with the results of the direction-averaged theory. Quantitative agreement can be obtained by appropriate choice of the averaging coefficients, e.g., 1) $g = 1/\sqrt{3}$, $g' = \frac{1}{2}$, 2) $g = g' = \frac{2}{3}$, or 3) $g = 0.6402$, $g' = 0.6146$.

The subsequent terms, however, do not agree with the direction-averaged theory. To the first approximation $\alpha_a = \alpha_{a0} \sim \sigma$, $\alpha_\tau \sim \sigma^2$, $\alpha_{a1} \sim \sigma^2$, all coefficients are proportional to Z, and $\alpha_{a1} \sim \lambda_r$, $\alpha_a \sim \lambda_r$, $\alpha_\tau \sim \lambda_r^2$.

For large v Eqs. (2.2) yield

$$\pm m_r = \frac{\gamma-1}{2\gamma}\zeta\left\{1 + \frac{1}{v^2} - \frac{\gamma^2+2\gamma+5}{8\gamma^2}\zeta^2 - \frac{\pi}{2v}\left(1 - \frac{\gamma^2+\gamma+1}{\gamma^2}\zeta^2\right) - \frac{1}{3v^4} - \frac{3(\pi^2+4)}{32}\frac{5\gamma^2+2\gamma+1}{\gamma^2}\frac{\zeta^2}{v^2} + \frac{7\gamma^4+12\gamma^3+18\gamma^2+28\gamma+63}{128\gamma^4}\zeta^4 + O(v^{-6}+\zeta^6)\right\},$$

$$\pm m_i = 1 + \frac{(\gamma-1)(\gamma+3)}{8\gamma^2}\zeta^2 - \frac{(\gamma-1)(\gamma+1)\pi}{4\gamma^2}\frac{\zeta^2}{v} + \frac{(\gamma-1)(3\gamma+1)}{32\gamma^2}(\pi^2+8)\frac{\zeta^2}{v^2} - \frac{(\gamma-1)(5\gamma^3+9\gamma^2+15\gamma+35)}{128\gamma^4}\zeta^4 + O(v^{-5}+\zeta^5).$$

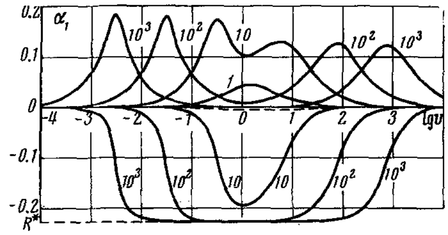

Fig. 5. Damping coefficient and dispersion factor r for $\gamma = 5/3$. The value of Z is indicated on each curve. The negative ordinate axis shows R = r − 1. R* denotes the value of R for $r = \gamma^{-1/2}$.

To a first approximation this agrees qualitatively with the approximate theory, with different meaning for ζ and Z. The damping coefficients are proportional to Z; $\alpha_a = \alpha_{a0} \sim \sigma^{-1}$, α_{a1} and α_τ are independent of σ; $\alpha_{a1} \sim \lambda_r^{-1}$, $\alpha_a \sim \lambda_r^{-1}$, α_{a0} is independent of λ_r. When $\gamma - 1 \equiv \delta \ll 1$, the waves are nearly adiabatic for arbitrary Z and v,

$$\pm m_r = a_1\delta(1 + b_1\delta) + O(\delta^3),$$

$$\pm m_i = 1 + a_2\delta(1 + b_2\delta) + O(\delta^3),$$

$$a_1 = {}^1/_2\beta(1+\beta^2)^{-1}, \quad a_2 = a_1\beta,$$

$$b_1 = {}^1/_2(1+\beta^2)^{-2}[\beta^2(3-\beta^2)K_1 + 2(\beta^4 - 2\beta^2 - 1)],$$

$$b_2 = {}^1/_4(1+\beta^2)^{-2}[2(3\beta^2-1)K_1 - (\beta^4 + 14\beta^2 + 5)],$$

$$K_1 = v(1+v^2)^{-1}K^{-1}. \qquad (2.7)$$

To a first approximation this yields

$$\alpha_1 = \alpha_{10} = \frac{\gamma-1}{2}\frac{\beta}{1+\beta^2}, \quad \alpha_\tau = \frac{\gamma-1}{2}\frac{\beta v}{1+\beta^2},$$

$$\alpha_1 = \frac{\gamma-1}{2}\frac{Zv}{3} \text{ when } v \ll 1, \quad \alpha_1 = \frac{\gamma-1}{2}\zeta \text{ when } v \gg 1. \quad (2.8)$$

When $\beta(v_m) \leq 1$ the coefficient $\alpha_1(v)$ has a single extremum–a maximum at the point $v = v_m$. When $\beta(v_m) > 1$ there are three extrema: a minimum at $v = v_m$ and a maximum, equal to $\delta/4$, at either side of the minimum, at v_{m1} and v_{m2}. The variables v_{m1} and v_{m2} are the roots of the equation $\beta = 1$ and are functions of the parameter Z. As Z increases from $Z = K^{-1}(v_m)$ to infinity v_{m1} decreases from v_m to zero, and v_{m2} increases from v_m to ∞. When $Z \gg 1$, $v_{m1}v_{m2} = 3$.

§3. Nearly isothermal waves. When $\beta \gg 1$ the speed of the waves differs from the isothermal speed of sound by a quantity of the order of β^{-2}. The roots of the characteristic equation can be represented by the series

$$\pm m_r = {}^1/_2(\gamma-1)[\beta_1 + b_3\beta_1^3 + b_5\beta_1^5 + O(\beta_1^7)],$$

$$\beta_1 = (K'Z)^{-1}, \quad K' = K(v'),$$

$$\pm m_i = \sqrt{\gamma}\left[1 - \frac{\gamma-1}{8\gamma}\beta_1^2(b_2 + b_4\beta_1^2) + O(\beta_1^6)\right], \qquad v' = \sqrt{\gamma}v,$$

$$b_2 = 5\gamma - 1 - 2K_2', \quad b_3 = 1 - 2\gamma + {}^1/_8K_2'\gamma^{-1}[3(5\gamma-1) + \alpha_3' - 4K_2'],$$

$$b_4 = \frac{1-5\gamma+2K_2'}{16\gamma}[(\gamma-1)(19\gamma+1) - 2(7\gamma-3)K_2' - 6\alpha_3'K_2' + 4K_2'^2] + 4(\gamma - K_2')b_3 - {}^1/_{12}\gamma^{-1}[6\gamma(\gamma-1)^2 - {}^1/_4(5-3v'^2)\alpha_3'^2K_2'],$$

$$K_2' = K_2(v'), \qquad \alpha_3' = \alpha_3(v'),$$

$$b_5 = \frac{K_2'-\gamma}{4\gamma}b_4 - \frac{b_3}{8\gamma}\{(\gamma-1)(7\gamma+1) + (3 - 11\gamma - 3\alpha_3' + 4K_2')K_2'\} + \frac{b_2}{64\gamma^2}\left\{3\gamma(\gamma-1)(7\gamma-3) + \frac{1}{2}K_2'\left[3(\gamma-1)(1-9\gamma) - \frac{65\gamma - 29 + 3(11\gamma+1)v'^2}{6(\gamma-1)}\alpha_3'^2 + \frac{3(3+v'^2)}{\gamma-1}\alpha_3'K_2'\right]\right\} - \frac{3v'^4 - 8v'^2 + 1}{768\gamma^2}K_2'\alpha_3'^3. \qquad (3.1)$$

For small v the condition $\beta \gg 1$ becomes $\zeta \ll 1$, and for large v it becomes $Zv \gg 1$, i.e., for large Z Eqs. (3.1) hold in the range $Z^{-1} \ll v \ll Z$. To a first approximation

$$\alpha_{10} = \frac{\gamma-1}{2}\beta_1, \quad \alpha_1 = \frac{\gamma-1}{2\sqrt{\gamma}}\beta_1, \quad \alpha_\tau = \frac{\gamma-1}{2}\beta_1 v,$$

$$v \ll 1, \quad m_r = \pm\frac{\gamma-1}{2Zv'}, \quad m_i = \pm\sqrt{\gamma}\left[1 - \frac{(\gamma-1)(5-\gamma)}{2\gamma^2Z^2v^2}\right],$$

$$\alpha_1 = \frac{\gamma-1}{2\gamma Zv}, \quad \alpha_\tau = \frac{\gamma-1}{2\sqrt{\gamma}Z},$$

$$v \gg 1, \quad m_r = \pm\frac{\gamma-1}{2}\frac{v'}{Z}, \quad m_i = \pm\sqrt{\gamma}\left[1 - \frac{(\gamma-1)(3\gamma+1)}{2\gamma}\frac{v'^2}{Z^2}\right],$$

$$\alpha_1 = \frac{\gamma-1}{2}\zeta_1, \quad \alpha_\tau = \frac{\gamma-1}{2}\zeta_1 v'. \qquad (3.2)$$

The coefficients α_{10}, α_1 reach a minimum:

$$v_{\min} = v_m / \sqrt{\gamma}, \quad \alpha_{10\min} = 2.175\,(\gamma-1)/Z,$$

$$\alpha_{1\min} = 2.175\,(\gamma-1)/(\sqrt{\gamma}Z). \qquad (3.3)$$

§4. Diffusely radiating waves. Let $Z \gg 1$, $\beta = O(1)$. This is equivalent to the conditions

$$Z \gg 1, \quad v \ll 1, \quad Zv = O(1);$$

$$Z \gg 1, \quad v \gg 1, \quad \zeta = O(1). \qquad (4.1)$$

For $v \ll 1$ we obtain

$$m_r = m_{r0} \mp 0.3(A_2^2 + B_2^2)^{-1}[m_{r0}(A_1A_2 + B_1B_2) + m_{i0}(A_1B_2 - A_2B_1)]v^2 + O(v^4), \quad m_i = m_{i0} \mp 0.3(A_2^2 + B_2^2)^{-1}[m_{i0}(A_1A_2 + B_1B_2) - m_{r0}(A_1B_2 - A_2B_1)]v^2 + O(v^4)],$$

$$A_1 = \gamma(m_{r0}^2 - m_{i0}^2) + (m_{r0}^4 - 6m_{r0}^2m_{i0}^2 + m_{i0}^4),$$

$$A_2 = \gamma + 2(m_{r0}^2 - m_{i0}^2),$$

$$B_1 = 2m_{r0}m_{i0}[\gamma + 2(m_{r0}^2 - m_{i0}^2)],$$

$$B_2 = 4m_{r0}m_0 - \gamma z_1, \qquad (4.2)$$

$$m_{r0} = \pm \frac{v_1}{\sqrt{2z_1}}, \quad m_{i0} = \pm \frac{v_2}{\sqrt{2z_1}}, \quad v_1 = \left[\frac{\gamma}{2}(a_4 - a_2 - z_1)\right]^{1/2},$$
$$v_2 = \left[\frac{\gamma}{2}(a_4 + a_2 + z_1)\right]^{1/2}, \quad a_1 = \left[(1 - z_1^2)^2 + \frac{4(2-\gamma)^2}{\gamma^2} z_1^2\right]^{1/2},$$
$$a_2 = [{}^1/_2 (a_1 - 1 + z_1^2)]^{1/2}, \quad a_3 = [{}^1/_2 (a_1 + 1 - z_1^2)]^{1/2}$$
$$a_4 = [1 + a_1 + z_1^2 + 2(a_2 z_1 - \nu a_3)]^{1/2}$$
$$(\nu = 1, \quad \text{if} \quad \gamma < 2, \quad \nu = -1, \quad \text{if} \quad \gamma > 2). \qquad (4.3)$$

To the first approximation these results coincide with those of the averaged theory, but in the averaged theory $z_1 = Z^\circ v^\circ$. The parameters Z and v appear only in the combination Zv. The coefficient α_T increases with z_1 from $(\gamma - 1)\ Zv^2\ /\ (6\gamma)$, proportional to σ^2 and λ_r^2, to $3(\gamma - 1)\ /\ (2\ \sqrt{\gamma}Z)$ which is independent of σ and λ_r. For equal z_1, $Z\alpha_T$ increases with increasing γ. For small z_1 to the first approximation $\alpha_{10} \sim \sigma$ and $\alpha_{10} \sim \lambda_r$, while for large z_1 $\alpha_{10} \sim \sigma^{-1}$ and $\alpha_{10} \sim \lambda_r^{-1}$, and

$$z_1 \ll 1, \quad \alpha_{10} = \alpha_1 = (\gamma - 1)\ Zv\ /\ (6\gamma);$$
$$z_1 \gg 1, \quad \alpha_{10} = \alpha_1 = 3(\gamma - 1)\ /\ (2\ \sqrt{\gamma}Zv). \qquad (4.4)$$

In the remaining range of z_1, for equal z_1 and γ $\alpha_1 < \alpha_{10}$, but the general form of the curves for $\alpha_{10}(z_1)$ and $\alpha_1(z_1)$ is similar. Each of these has one extremum—a maximum

$$z_{1\max} = 1, \quad \alpha_{10\max} = \tfrac{1}{2}(2\sqrt{\gamma} - \gamma - \sqrt{\gamma(2-\gamma)})^{1/2}$$
$$\alpha_{1\max} = (1 - \sqrt{2-\gamma})(1 + \sqrt{\gamma})^{-1} \qquad (4.5)$$

This case corresponds to the diffusion approximation for radiation. Assuming that the radiative transfer can be represented by radiative thermal conductivity, we can replace the equation of radiative transfer in (1.1) by the relation $H = -k_r \partial T / \partial x$ and thus obtain (4.3) instead of the characteristic equation (1.3).

§5. Emitting waves. In the case $Z \gg 1$, $\zeta = 0(1)$ we have

$$m_r = m_{r0} \mp \frac{\pi(\gamma-1)(\gamma-\zeta^2)\zeta}{4(1+\zeta^2)(\gamma^2+\zeta^2)} v^{-1} + \frac{(\gamma-1)(m_{i0}A - m_{r0}B)\zeta}{2\gamma(1+\zeta^2)^2} v^{-2} + \ldots,$$
$$m_i = m_{i0} \mp \frac{\pi(\gamma^2-1)\zeta^2}{4(1+\zeta^2)(\gamma^2+\zeta^2)} v^{-1} - \frac{(\gamma-1)(m_{r0}A + m_{i0}B)\zeta}{2\gamma(1+\zeta^2)^2} v^{-2} + \ldots,$$
$$A = 1 - \zeta^2 + a\zeta[4\zeta^4 + (3\gamma^2 - 2\gamma - 11)\zeta^2 - (9\gamma^2 + 2\gamma - 1)],$$
$$B = -2\zeta + a[(\gamma+1)\zeta^4 + (9\gamma^2 - 5)\zeta^2 - \gamma(3\gamma+1)], \qquad (5.1)$$
$$a = \frac{\pi^2\zeta}{16(\gamma^2+\zeta^2)(1+\zeta^2)},$$
$$m_{r0} = \pm\left\{\frac{\gamma[\sqrt{\gamma^2\zeta_1^4 + (\gamma^2+1)\zeta_1^2 + 1} - \gamma\zeta_1^2 - 1]}{2(1+\gamma^2\zeta_1^2)}\right\}^{1/2},$$
$$m_{i0} = \pm\left\{\frac{\gamma[\sqrt{\gamma^2\zeta_1^4 + (\gamma^2+1)\zeta_1^2 + 1} + \gamma\zeta_1^2 + 1]}{2(1+\gamma^2\zeta_1^2)}\right\}^{1/2}, \qquad (5.2)$$

To the first approximation we have purely emitting waves [1], which depend only on the emission of radiation. To the first approximation the expressions agree with the results based on the direction-averaged equation of radiative transfer. In the limiting cases

$$\zeta_1 \ll 1, \quad \alpha_{10} = \frac{\sqrt{\gamma}}{2}(\gamma-1)\zeta_1, \quad \alpha_1 = \frac{1}{2}(\gamma-1)\zeta_1,$$
$$\alpha_\tau = \frac{\sqrt{\gamma}}{2}(\gamma-1)Z\zeta_1^2, \quad \alpha_{a1} = \frac{\gamma-1}{2}\sqrt{\gamma}\zeta_1\frac{\sigma}{c_0},$$
$$\zeta_1 \gg 1, \ \alpha_{10} = \frac{\gamma-1}{2\gamma}\zeta, \quad \alpha_1 = \frac{\gamma-1}{2\gamma}\zeta, \quad \alpha_\tau = \frac{\gamma-1}{2\gamma}Z,$$
$$\alpha_{a1} = \frac{\gamma-1}{2\gamma}\zeta\frac{\sigma}{c_0}. \qquad (5.3)$$

The variables ζ_1, α_{10}, and α_1 have single extrema—maxima

$$\zeta_{1\max} = \frac{1}{\gamma}\left(\frac{3\gamma+1}{\gamma+3}\right)^{1/2},$$
$$\alpha_{10\max} = \frac{1}{2}\frac{\gamma-1}{\sqrt{2(\gamma+1)}},$$
$$\zeta_{1\max} = \frac{1}{\sqrt{\gamma}}, \quad \alpha_{1\max} = \frac{\sqrt{\gamma}-1}{\sqrt{\gamma}+1}. \qquad (5.4)$$

The damping coefficient increase with increasing γ.

This case corresponds, to the first approximation, to the Newtonian theory of wave motion in purely emitting media, without absorption [2, 3]. Assuming that the heat transfer is due only to the deviation of the radiative emission from its equilibrium value, one can replace B − J in the right side of the equation of radiative transfer by $B - B_0$ to obtain (5.2) instead of (1.3).

§6. Wave attenuation in an emitting-absorbing gas. For small Z the portion of energy transferred by radiation is small, the process is nearly adiabatic, the damping of the pressure waves is weak, and the waves propagate at the Laplacian speed of sound. This case is described in §2. In the case of large optical thicknesses the interior of each wave reaches radiative equilibrium and in the case of large frequencies the waves approach a frozen state. For moderate values of the optical thickness of the wavelength one obtains maximum values of the damping coefficients α_{10}, α_1 and minimum speed. When Z increases to 4–5 the damping coefficients increase, but the general form of the dependence on v remains the same, as can be seen in Figs. 2–4, which are based on numerical solutions of (1.3). Further increase in Z results in a qualitative change of the curves α_{10} and α_1 in the range v = O(1): first, two additional inflection points appear to the right of the maximum and then a minimum and a maximum are formed. This change becomes more pronounced the higher is the value of γ. For small $\gamma - 1$ the original maximum ($Z \ll 1$) at v = 1 splits with increasing Z into two extrema, the one near v = 1 becoming a minimum. For moderate Z the behavior of the waves in the case of small and large v is described by (2.2), and for other values of Z it is described by (1.3). There appears a dependence of the quantity $\alpha_T Z^{-1}$ on Z: with increasing Z the curve moves toward large v and the coefficient $\alpha_T(v)$ increases as a monotone function of Z and increases with increasing γ. The coefficients α_{10}, α_1 increase with increasing Z for arbitrary γ only for small Z. For moderate and large Z this rule remains valid only in the regions $z_1 \ll 1$ and $\zeta \ll 1$.

For $Z \gg 1$ and small or large v, when $z_1 \ll 1$ or $\zeta \ll 1$, the waves are described by the equations of §2. In the range of moderate z_1 and ζ the proper equations and results are those of §4 and §5. When β is large, the waves are described by the results of §3. These limiting equations cover the full range of v. Thus, for sufficiently large Z the coefficients α_{10}, α_1 have two maxima and one minimum each.

The magnitudes of the maxima depend only on γ, the position of the first of these depends only on Z, and the position of the other one depends on γ and Z. Two "relaxation" times are significant. The minima depend on γ and Z, and their positions depend only on γ.

The coefficient $\alpha_T Z^{-1}$ is a monotone function of v and increases from 0 to $(\gamma - 1)/2\gamma$ and is almost the same for all large values of Z. The curves α_{10}, α_1, α_T lie closer to the horizontal axis as γ decreases. For a given value of γ the left maximum of α_{10} or α_1 is larger than the right maximum. All extrema increase with increasing γ in such a way that for a given value of Z the left maxima of α_{10} or α_1 are at the same v, independent of v, whereas the minima and the right maxima move to the left, the right maximum of $\alpha_{10}(v)$ being to the left of the right maximum of $\alpha_1(v)$.

The general form of the variation of the true damping coefficient and the wave speed for large Z is shown in Fig. 5. We can distinguish four regions:

1) In the range $v \ll Z^{-1}$ the transfer of heat inside the wave is by a radiative-conductivity mechanism, but the relative magnitude of this heat flux is small, so that the conditions are nearly adiabatic. The waves propagate at the adiabatic speed of sound and decay slowly.

2) In the region $v = O(Z^{-1})$ the diffusive radiative heat transfer increases with decreasing optical thickness of the wave, until nearly iso-

thermal conditions are reached. Smaller optical thicknesses of the waves correspond to stronger radiative approach to equilibrium. The wave speed decreases from the adiabatic wave speed to the isothermal wave speed. The coefficients α_{10} and α_1 reach maximum values in the middle of the range and are small near the boundaries of the range.

3) In the region $Z^{-1} \ll v \ll Z$ the process is isothermal: the temperature has time to reach equilibrium during one period of the wave. The damping coefficients are small and the wave speed is equal to the Newtonian speed of sound. In the range $v = O(1)$ all variables reach minimum values.

4) When $v = O(Z)$ the optical thickness is so small that the radiation emitted by a wave is not reabsorbed. The waves emit, the wave speed increases from the isothermal speed of sound to the adiabatic speed of sound, and the damping coefficients again reach a maximum in the middle of the region.

5) When $v \ll Z$ the waves become so short, and the frequency so large, that the period of one oscillation is insufficient for the transfer of energy in the wave and the conditions are adiabatic.

The basic cases of wave propagation can also be characterized by the value of the variable β. If $\beta \ll 1$, then, as can be seen from (1.5), the heat flux is small and the waves are weakly damped, nearly adiabatic sound waves and can be described by the equations of §2. If, on the other hand, $\beta \gg 1$, which is possible only for large Z, then the waves are nearly isothermal and can be described by the equations of §3. Finally, if $\beta = O(1)$, the wave speed cannot be close either to the adiabatic or the isothermal speed of sound and lies somewhere between these. Three cases are then possible:

1) $v = O(1)$, $Z = O(1)$, and one must consider the full characteristic equation; 2) $v \ll 1$, $z_1 = O(1)$, the dissipation of energy takes place by a diffusive mechanism and §4 holds; 3) when $v \gg 1$, $\zeta = O(1)$ §5 holds. The radiation emitted by a wave has no time to be reabsorbed. This case corresponds to Stokes' theory.

In terms of relaxation theory there are three relaxation times in the regions $\vartheta = O(t_{ra})$, $\vartheta = O(t_{re})$, $t_{ra}^2 = O(\vartheta t_{re})$, and for $Z \ll 1$ only the first of these is significant. These times were defined in terms of v, Z, γ above.

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